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DISCUSSION PAPER

Leader-Dependent Hedonic Majority Games

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Abstract We introduce leader-dependent hedonic games, where each player has a preference over pairs of a coalition and its leader. We focus on leader-dependent hedonic games for a majority-game case. Based on a leader-dependent hedonic majority game, we define a noncooperative bargaining game as follows: a player proposes a coalition, and each player in the proposed coalition accepts or rejects the proposal; if all players accept it, the proposed coalition is formed and the proposer becomes its leader, and otherwise, the bargaining proceeds to the next round with a similar procedure. We characterize stationary subgame perfect equilibria: if players are impatient, all proposers immediately successfully form a minimal winning coalition of relatively impatient players; if players are patient, relatively impatient proposers immediately successfully form a minimal wining coalition of relatively impatient players, but relatively patient proposers fail to form a coalition, which implies that bargaining delays occur; if players are furthermore patient, a minimal winning coalition of relatively patient players may be formed, and multiple equilibria exhibiting delays may exist.

Keywords: Hedonic games; Majority games; Heterogeneous time preferences; Bargaining delay; Multiple equilibria
JEL classification codes: C78; D72

1 Introduction

We extend hedonic games as players' preferences may depend on the leaders of coalitions. In a hedonic game, each player has a preference over coalitions. However, in the leader-dependent hedonic game defined in this paper, each player has a preference over coalitions with leaders. In reality, coalitions often have a leader, and each coalition member's well-being depends on the identity of the leader because the leader substantially influences decisions within the coalition. For example, if two coalition governments have the same members but different prime ministers, generically each government member is not indifferent between the two governments. We focus on a typical q-majority-game case: only coalitions consisting of q or more players are admissible, and each player's utility from a coalition with a leader is his/her equilibrium payoff in bargaining over allocations within this coalition with its leader being the first proposer. We assume that players have different time preferences.

Based on a leader-dependent hedonic majority game, we define a noncooperative sequential bargaining game. In the noncooperative game, a player proposes a coalition, and each player in the proposed coalition accepts or rejects the proposal; if all players accept it, the proposed coalition is formed and the proposer becomes its leader, and otherwise, bargaining proceeds to the next round where the first rejecter becomes a proposer and a similar procedure takes place. This noncooperative game is interpreted as players committing to forming a coalition and then bargaining over allocations within this coalition or players not being able to commit to allocations in the coalition formation phases and thus bargaining over allocations after a coalition is formed.

We characterize stationary subgame perfect equilibria (SSPEs) in the noncooperative game. (i) If players are impatient, all proposers immediately successfully form a minimal winning coalition consisting of relatively impatient players; (ii) if players are patient, relatively impatient proposers immediately successfully form a minimal wining coalition of relatively impatient players, but relatively patient proposers fail to form a coalition, which implies that bargaining delays occur; (iii) if players are furthermore patient, a minimal winning coalition of relatively *patient* players may be formed, and multiple SSPEs exhibiting delays may exist. The delay in (ii) would not occur if each player would be allowed to propose a singleton consisting of himself/herself, so it is considered somewhat nominal delay. However, the delay in (iii) would occur even if each player could propose a singleton, so it is considered real delay.

Leader-dependency causes bargaining delays to be more likely to occur. Bloch and Diamantoudi (2011) investigate a noncooperative sequential bargaining game based on a leader-*independent* hedonic game. They show that if the restriction of the hedonic game to any subset of players has a singleton core, there exists an SSPE and any SSPE exhibits no delay and generates a unique core coalition structure. If we modify our leader-dependent hedonic majority game by letting preferences not depend on leaders,[1] a coalition structure such that the least patient q players form a coalition and the other players respectively form a singleton is a unique core coalition structure, so SSPEs in the noncooperative game based on this modified leader-independent hedonic game exist, exhibit no delay and have a unique payoff tuple. The reason for the difference in the delay between leader-independent preferences and leader-dependent ones is as follows: under leader-independent preferences, each player is indifferent between a coalition formed when he/she is a proposer (he/she becomes the leader of this coalition) and the same coalition formed when he/she is a responder (he/she does not become the leader of this coalition); under leader-dependent preferences, he/she prefers the former to the latter; thus, under leader-dependent preferences, each player has a stronger incentive to become proposer by rejecting a proposal, so delays are more likely to occur.

Not allowing allocations to be decided in the coalition formation phase causes bargaining delays to be more likely to occur. Kawamori (2013) investigates a noncooperative sequential bargaining game similar to our game except for that players simultaneously bargain over both coalitions and allocations. He shows that there exists an

[1] For example, we can modify the bargaining over allocations by letting the first proposer be randomly selected regardless of the identity of leader.

SSPE any SSPE exhibits no delay and SSPEs have a unique payoff tuple.[2] In Kawamori (2013), each patient player gives impatient players such large shares of the surplus that they accept the proposal; in our paper, each patient player cannot do so, and thus, delays may occur. Meanwhile, if an impatient player is the first proposer and players are patient (case (ii) above), the game in Kawamori (2013) and that in our paper have the same SSPE payoff tuple such that a more patient player has a larger SSPE payoff among the q least patient players and the other players' SSPE payoffs are zero.

The remainder of this paper is organized as follows: Section 2 describes the model; Section 3 characterizes SSPEs; Appendix gives all propositions proofs.

2 Model

2.1 Leader-dependent hedonic games

We define leader-dependent hedonic games.

Definition 1. A *leader-dependent hedonic game* is a triple $\left(N, \mathcal{S}, (\succsim_i)_{i \in N}\right)$ such that N is a finite set, $\mathcal{S}$ is a subset of $\left\{(S, j) \in 2^N \times N \mid j \in S\right\} =: 2^N_*$, for any $i \in N$, $\succsim_i$ is a preference relation (total and transitive binary relation) over $\{(S, j) \in \mathcal{S} \mid S \ni i\} =: \mathcal{S}_i$, and for any $i \in N$ and any $(S, j) \in \mathcal{S}_i$, $(S, j) \succsim_i (S, i)$ implies $j = i$.

N is the set of players, $\mathcal{S}$ is the set of admissible pairs of a coalition and its leader, $\succsim_i$ is player i's preference relation over admissible pairs of a coalition to which he/she belongs and its leader, and each player prefers his/her being a leader to another member being the leader. It is natural that coalitions have leaders and identities of them affect coalition members' well-beings. Let $n = |N|$.

We focus on leader-dependent hedonic games in a majority-game case (including a supermajority-game case). Assume that there exists $q \in \mathbb{N}$ with $\frac{n}{2} < q \leq n$ such that $\mathcal{S} = \left\{(S, j) \in 2^N_* \mid |S| \geq q\right\}$. Coalitions of q or more players are wining coalitions.

Furthermore, we confine the scope of this paper to typical leader-dependent hedonic majority games, where after a coalition is formed, coalition members bargain

[2] Eraslan (2002) shows the same thing for legislative bargaining under the random-proposer protocol.

over allocations of a fixed surplus. Assume that $\succsim_i$ is represented by utility function $(S,h) \mapsto x_i^{hS}$, where x_i^{hS} is defined and characterized as follows. For any $(S,h) \in 2_*^N$ such that $|S| \geq q$, let $x^{hS} \in \mathbb{R}_+^S$ such that $x_h^{hS} = \frac{\frac{1}{1-\delta_h}}{1+\sum_{j\in S}\frac{\delta_j}{1-\delta_j}}$, and for any $i \in S \setminus \{h\}$, $x_i^{hS} = \frac{\frac{\delta_i}{1-\delta_i}}{1+\sum_{j\in S}\frac{\delta_j}{1-\delta_j}}$, where $\delta \in (0,1)^N$. x^{hS} is a unique SSPE payoff tuple in the extensive form game defined as follows.[3] Note that this assumption satisfies the property of $\succsim_i$ in Definition 1. There are infinite rounds indexed by S: an index $i \in S$ is the proposer in each round. The game begins with a round with h, who is the leader of S. In any round with i, bargaining proceeds as follows.

1. Player i proposes an allocation for coalition S, i.e., an $x \in \mathbb{R}_{\geq 0}^S$ such that $\sum_{j\in S} x_j = 1$.

2. Each player in $S \setminus \{i\}$ accepts or rejects this proposal sequentially according to some predetermined order until one player rejects it or all players accept it.

- If player j rejects it, the game proceeds to a round with j.

- If all players in $S \setminus \{i\}$ accept it, the proposed allocation x is implemented, and the game ends.

If an agreement x is achieved in the tth round, player i's payoff is x_i^{t-1}; if no agreement is achieved, any player's payoff is 0.

2.2 Extensive form game

We define an extensive form game G based on the leader-dependent hedonic game $(N, \mathcal{S}, (\succsim_i)_{i\in N})$ as follows. N is the set of players. Let $\mathcal{Q} := \{S \in 2^N \mid |S| = q\}$. For any $i \in N$, let $\mathcal{Q}_i := \{S \in \mathcal{Q} \mid i \in S\}$. There are infinite rounds indexed by N: an index $i \in N$ is the proposer in each round. The game begins with a round with some predetermined element f in N. In any round with $i \in N$, bargaining proceeds as follows.

[3] This is well known in the literature. For example, see the unanimity-rule case in Kawamori (2013).

1. Player i proposes a minimal winning coalition to which he/she belongs, i.e., an $S \in \mathcal{Q}_i$.

2. Each player in $S \setminus \{i\}$ accepts or rejects this proposal sequentially according to some predetermined order until one player rejects it or all players accept it.

- If player j rejects it, the game proceeds to another round with j.

- If all players in $S \setminus \{i\}$ accept it, coalition S forms and player i becomes a leader of S, and the game ends.

Let $\alpha \in (0,1)^N$: for any $i \in N$, α_i is player i's discount factor. If an agreement S is achieved in the tth round with proposer i, the payoff of player $j \in S$ is $\alpha_j^{t-1} x_j^{iS}$, and that of player $j \notin S$ is zero; if no agreement is achieved, any player's payoff is 0.

We consider pure strategies. A strategy tuple in G is *stationary* if each player's proposal is the same at all his/her proposing nodes and each player's response is the same at all his/her responding nodes at which the same proposal is offered. A *stationary subgame perfect equilibrium (SSPE)* in G is a subgame perfect equilibrium (SPE) that is stationary.

3 Results

Assume that $N = \{1, \dots, n\}$. Further, assume that for any $i, j \in N$, if $i < j$, then $\alpha_i < \alpha_j$, and $\delta_i < \delta_j$. Note that $\alpha_i < \delta_i$ ($\alpha_i > \delta_i$) for any $i \in N$ is interpreted as bargaining over coalitions goes with more (less) friction than bargaining over allocations. Let $W = \{i \in N \mid i \leq q\}$: W is the minimal wining coalition of the least patient q players. For any $i \in W$, let $W_i = W$; for any $i \in N \setminus W$, let $W_i = \{j \in N \mid j \leq q-1\} \cup \{i\}$: W_i is the minimal wining coalition of player i and the least patient $q-1$ players other than player i. As the other coalition members are less patient, player i has a stronger relative bargaining power; thus, W_i is the most preferable coalition for player i.

Proposition 1 states that if α_i is sufficiently small for any i, in any SSPE, the first proposer's minimal wining coalition of impatient players is immediately formed.

Proposition 1. *Suppose that for any $i \in N$, $\alpha_i < \delta_i \min_{S \in \mathcal{Q}_i \setminus \{W_i\}} \frac{x_i^{iS}}{x_i^{iW_i}}$. Then, there exists an SSPE, and in any SSPE, player f's proposal W_f is accepted immediately.*

Under a sufficiently small α_i, the cost of rejection for each responder is so high that he/she accepts the most preferable coalition for each proposer; thus, each proposer proposes the most preferable coalition for him/her.

Proposition 2 states that if α_i is sufficiently large but smaller than δ_i for any i, in any SSPE, if the first proposer is among the least patient q players, a minimal wining coalition of impatient players is immediately formed; otherwise, it is formed with delay or no coalition is formed.

Proposition 2. *Suppose that for any $i \in N$, $\delta_i \max_{S \in \mathcal{Q}_i \setminus \{W_i\}} \frac{x_i^{iS}}{x_i^{iW_i}} < \alpha_i < \delta_i$. Then, there exists an SSPE, and in any SSPE,*

- *if $f \in W$, player f's proposal W is accepted immediately;*
- *otherwise, proposal W is accepted in some subsequent period, or no proposal is accepted.*

Remark 1. If the first proposer is so patient that he/she is not in W, there exists an SSPE and any SSPE exhibits delays. In the proof of existence, we define a strategy tuple and show that it is an SSPE. In this SSPE, rejection is infinitely repeated if $n - q \geq 2$, player n is the first proposer and players n and $n-1$ are the first and second responders, respectively.

The least patient q players mutually propose and accept W: by rejecting W, each i of the least patient q players can become a leader (his/her share is multiplied by $\frac{1}{\delta_i} > 1$), whereas bargaining exhibits delays (his/her payoff is discounted by $\alpha_i < 1$); because $\alpha_i < \delta_i$, the latter dominates the former; thus, he/she accepts W; hence, each of the least patient q players proposes the most preferable W. Proposal $S \neq W$ of each of the most patient $n - q$ players is rejected: by rejecting S, each i of the least patient q players can become a leader of the most preferable W instead of S, whereas bargaining exhibits delays (his/her payoff is discounted by α_i); α_i is so large that the latter is dominated by the former ($\alpha_i > \delta_i \max_{S \in \mathcal{Q}_i \setminus \{W_i\}} \frac{x_i^{iS}}{x_i^{iW_i}}$); thus, he/she rejects S.

Proposition 3 implies that in the three-player case, if α_i is slightly larger than δ_i for any i, there exists an SSPE such that the least patient player proposes the coalition with the most patient player and the coalition is formed; there also exists another SSPE such that the second least patient player proposes the coalition with the most patient player and the coalition is formed.

Proposition 3. *Suppose that $n = 3$ and $q = 2$. Suppose that for any $i \in \{1,2\}$, $\delta_i < \alpha_i < \min\left\{\delta_i \frac{x_i^{i\{1,2\}}}{x_i^{i\{i,3\}}}, \frac{1}{\delta_i}\frac{x_i^{i\{i,3\}}}{x_i^{i\{1,2\}}}\right\}$, and $\delta_3 < \alpha_3 < \sqrt{\frac{x_3^{3\{2,3\}}}{x_3^{3\{1,3\}}}}$. Let $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$. Then, there exists an SSPE such that*

- *if $f = i$, player i's proposal $\{1,2\}$ is accepted immediately;*
- *if $f = j$, player j's proposal $\{j,3\}$ is accepted immediately;*
- *otherwise, player j's proposal $\{j,3\}$ is accepted in the second period.*

Remark 2. For any $i \in \{1,2\}$, if $\delta_i < \sqrt{\frac{x_i^{i\{i,3\}}}{x_i^{i\{1,2\}}}}$, then $\delta_i < \min\left\{\delta_i \frac{x_i^{i\{1,2\}}}{x_i^{i\{i,3\}}}, \frac{1}{\delta_i}\frac{x_i^{i\{i,3\}}}{x_i^{i\{1,2\}}}\right\}$: if δ_i is sufficiently small, there exists α_i that satisfies the supposition of Proposition 3. If δ_3 is sufficiently small, there exists α_3 that satisfies the supposition of Proposition 3.

Remark 3. There exist multiple SSPEs such that different coalitions are formed and thus different payoff tuples are realized as follows.

- If $f = 1$,
  - in an SSPE ($i = 1$), player 1's proposal $\{1,2\}$ is accepted immediately;
  - in another SSPE ($i = 2$), player 1's proposal $\{1,3\}$ is accepted immediately.
- If $f = 2$,
  - in an SSPE ($i = 1$), player 2's proposal $\{2,3\}$ is accepted immediately;
  - in another SSPE ($i = 2$), player 2's proposal $\{1,2\}$ is accepted immediately.
- If $f = 3$,
  - in an SSPE ($i = 1$), player 2's proposal $\{2,3\}$ is accepted in the second period;

– in another SSPE ($i = 2$), player 1's proposal $\{1, 3\}$ is accepted in the second period.

Features of the SSPE with $i = 1$ and $j = 2$ in this proposition are that player 1 rejects $\{1, 2\}$ and player 2 proposes not $\{1, 2\}$ (the most preferable coalition) but $\{1, 3\}$. In this SSPE, player 1's proposal $\{1, 2\}$ is accepted; thus, by rejecting $\{1, 2\}$, player 1 can become a leader (his/her share is multiplied by $\frac{1}{\delta_i}$), whereas bargaining exhibits delays (his/her payoff is discounted by α_i); because $\alpha_i > \delta_i$, the latter is dominated by the former; thus, he/she rejects $\{1, 2\}$. Thus, by proposing $\{1, 2\}$, player 2 fails to become the leader, and bargaining exhibits delays; hence, player 2 proposes not $\{1, 2\}$ but $\{1, 3\}$. In other words, rather than chooses an impatient coalition partner and does not become a leader, player 2 chooses a patient coalition partner and becomes a leader. The same thing occurs in the other SSPE in the proposition if player 1 is replaced by player 2. These two SSPEs cause different payoff tuples.

Appendix

Let $\triangleleft$ be the order of responses.[4] Let $\triangleright := \{(i,j) \mid (j,i) \in \triangleleft\}$. For any stationary strategy tuple σ in G, any $i,j,k \in N$ with $i \neq j$ any $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$, any $T \in \mathcal{Q}_k$ and any $b \in \{0,1\}$, let $\sigma_i(i)$ be player i's proposal specified by σ, $\sigma_i(j,S)$ be player i's response specified by σ to player j's proposal S, $u_i(\sigma|k)$ be player i's payoff by σ in any subgame starting with player k's proposing node, $u_i(\sigma|j,S)$ be player i's payoff by σ in any subgame starting with player i's responding node following player j's proposal S, $u_i(\sigma,T|k)$ be player i's payoff by player k's one-shot deviation from σ to T in any subgame starting with player k's proposing node and $u_i(\sigma,b|j,S)$ be player i's payoff by player i's one-shot deviation from σ to b in any subgame starting with player i's responding node following player j's proposal S.

A Lemmas

Lemma 1. *Let $i \in N$. Let $S \in \mathcal{Q}_i \setminus \{W_i\}$. Then, $x_i^{iW_i} > x_i^{iS}$.*

Proof. Let $\pi : W_i \setminus \{i\} \to S \setminus \{i\}$ such that π is the order isomorphism. By the mathematical induction, show that for any $j \in W_i \setminus \{i\}$, $j \leq \pi(j)$. $\min(W_i \setminus \{i\}) = \min(N \setminus \{i\}) \leq \pi(\min(W_i \setminus \{i\}))$. Let $j \in (W_i \setminus \{i\}) \setminus \{\max(W_i \setminus \{i\})\}$. Suppose that $j \leq \pi(j)$ (induction hypothesis). Let $k := \min\{l \in W_i \setminus \{i\} \mid l > j\}$. Because π is the order isomorphism, $\pi(j) < \pi(k)$. Thus, by the induction hypothesis, $j < \pi(k)$. Hence, because $k = \min\{l \in W_i \setminus \{i\} \mid l > j\} = \min\{l \in N \setminus \{i\} \mid l > j\}$, $k \leq \pi(k)$. Therefore, for any $j \in W_i \setminus \{i\}$, $j \leq \pi(j)$. Thus, for any $j \in W_i \setminus \{i\}$, $\delta_j \leq \delta_{\pi(j)}$, and thus, $\frac{1}{1-\delta_j} \leq \frac{1}{1-\delta_{\pi(j)}}$. Because $S \neq W_i$, for some $j \in W_i \setminus \{i\}$, $j \neq \pi(j)$; thus, $j < \pi(j)$; hence, $\delta_j < \delta_{\pi(j)}$; therefore, $\frac{1}{1-\delta_j} < \frac{1}{1-\delta_{\pi(j)}}$. Thus, $\sum_{j \in W_i \setminus \{i\}} \frac{1}{1-\delta_j} < \sum_{j \in W_i \setminus \{i\}} \frac{1}{1-\delta_{\pi(j)}} = \sum_{j \in S \setminus \{i\}} \frac{1}{1-\delta_j}$. Hence, $\sum_{j \in W_i} \frac{1}{1-\delta_j} < \sum_{j \in S} \frac{1}{1-\delta_j}$. Therefore, $x_i^{iW_i} > x_i^{iS}$. □

Lemma 2. *Let $i,j \in N$ and σ be a stationary strategy tuple. Then, $u_i(\sigma|j) \leq x_i^{iW_i}$.*

[4] For any coalition S proposed by any player i and any distinct $j,k \in S \setminus \{i\}$, player j responds earlier than player k if and only if $j \triangleleft k$.

Proof. For some $t \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, some $S \in \mathcal{Q}_i$ and some $k \in S$, $u_i(\sigma|j) = \alpha_i^t x_i^{kS} \leq x_i^{iS}$, and thus, by Lemma 1, $u_i(\sigma|j) \leq x_i^{iW_i}$. □

B Proof of Proposition 1

Existence Let σ be the stationary strategy tuple such that for any $i \in N$, any $j \in N \setminus \{i\}$ and any $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$, $\sigma_i(i) = W_i$ and $\sigma_i(j, S) = 1$. Let $i \in N$. (i) $\sigma_i(i) = W_i$, and for any $j \in W_i \setminus \{i\}$, $\sigma_j(i, W_i) = 1$; thus, $u_i(\sigma|i) = x_i^{iW_i}$. Let $S \in \mathcal{Q}_i$. Because for any $j \in S \setminus \{i\}$, $\sigma_j(i, S) = 1$, $u_i(\sigma, S|i) = x_i^{iS}$. Thus, by Lemma 1, $u_i(\sigma|i) \geq u_i(\sigma, S|i)$. (ii) Let $j \in N \setminus \{i\}$ and $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$. For any $k \in S \setminus \{j\}$, $\sigma_k(j, S) = 1$; thus, $u_i(\sigma|j, S) = x_i^{jS} = \delta_i x_i^{iS}$. $u_i(\sigma, 0|j, S) = \alpha_i u_i(\sigma|i) = \alpha_i x_i^{iW_i}$. Thus, because $\alpha_i < \delta_i \frac{x_i^{iS}}{x_i^{iW_i}}$, $u_i(\sigma|j, S) \geq u_i(\sigma, 0|j, S)$. Thus, by the one-shot deviation principle, σ is an SPE.

Characterization Let σ be an SSPE.

Lemma 3. *Let $i \in N$ and $S \in \mathcal{Q}_i$. Then, for any $j \in S \setminus \{i\}$, $\sigma_j(i, S) = 1$.*

Proof. By the mathematical induction, show that for any $j \in S \setminus \{i\}$, $\sigma_j(i, S) = 1$. Let $j \in S \setminus \{i\}$. Suppose that for any $k \in S \setminus \{i\}$ with $k \triangleright j$, $\sigma_k(i, S) = 1$ (induction hypothesis). Then, $u_j(\sigma, 1|i, S) = x_j^{iS} = \delta_j x_j^{jS}$. By Lemma 2, $u_j(\sigma, 0|i, S) = \alpha_j u_j(\sigma|j) \leq \alpha_j x_j^{jW_j}$. Thus, because $\alpha_i < \delta_j \frac{x_j^{jS}}{x_j^{jW_j}}$, $u_j(\sigma, 1|i, S) > u_j(\sigma, 0|i, S)$. Hence, $\sigma_j(i, S) = 1$. Therefore, for any $j \in S \setminus \{i\}$, $\sigma_j(i, S) = 1$. □

Lemma 4. *Let $i \in N$. Then, $\sigma_i(i) = W_i$.*

Proof. By Lemma 3, for any $S \in \mathcal{Q}_i$, $u_i(\sigma, S|i) = x_i^{iS}$. Thus, by Lemma 1, $\sigma_i(i) = W_i$. □

By Lemma 4, $\sigma_f(f) = W_f$. By Lemma 3, for any $i \in W_f \setminus \{f\}$, $\sigma_i(f, W_f) = 1$. □

C Proof of Proposition 2

Existence Let σ be the stationary strategy tuple such that

- for any $i \in W$, any $j \in N \setminus \{i\}$ and any $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$, $\sigma_i(i) = W$, and $\sigma_i(j, S) = 1$ if and only if $S = W$;

- for any $i \in N \setminus W$, any $j \in N \setminus \{i\}$ and any $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$, $\sigma_i(i) = \{k \in N \mid k \geq n - q + 1\}$, and $\sigma_i(j, S) = 1$ if and only if $\{k \in S \cap W \setminus \{j\} \mid k \triangleright i\} = \emptyset$.

Let $i \in W$. (i) $\sigma_i(i) = W$, and for any $j \in W \setminus \{i\}$, $\sigma_j(i, W) = 1$; thus, $u_i(\sigma|i) = x_i^{iW}$. Let $S \in \mathcal{Q}_i$. If for any $j \in S \setminus \{i\}$, $\sigma_j(i, S) = 1$, by Lemma 1, $u_i(\sigma, S|i) = x_i^{iS} \leq x_i^{iW}$; otherwise, by Lemma 2, for some $j \in N \setminus \{i\}$, $u_i(\sigma, S|i) = \alpha_i u_i(\sigma|j) \leq x_i^{iW}$. Thus, $u_i(\sigma|i) \geq u_i(\sigma, S|i)$. (ii) Let $j \in N \setminus \{i\}$ and $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$. Suppose that $S = W$. For any $k \in S \setminus \{j\}$, $\sigma_k(j, S) = 1$; thus, $u_i(\sigma|j, S) = x_i^{jS} = x_i^{jW} = \delta_i x_i^{iW}$. $u_i(\sigma, 0|j, S) = \alpha_i u_i(\sigma|i) = \alpha_i x_i^{iW}$. Thus, because $\delta_i > \alpha_i$, $u_i(\sigma|j, S) \geq u_i(\sigma, 0|j, S)$. Suppose that $S \neq W$. $\sigma_i(j, S) = 0$; thus, $u_i(\sigma|j, S) = \alpha_i u_i(\sigma|i) = \alpha_i x_i^{iW}$. If for any $k \in S \setminus \{j\}$ such that $i \triangleleft k$, $\sigma_k(j, S) = 1$, because $\delta_i \max_{T \in \mathcal{Q}_i \setminus \{W_i\}} \frac{x_i^{iT}}{x_i^{iW_i}} < \alpha_i$, $u_i(\sigma, 1|j, S) = x_i^{jS} = \delta_i x_i^{iS} \leq \alpha_i x_i^{iW}$; otherwise, by Lemma 2, for some $k \in N \setminus \{i\}$, $u_i(\sigma, 1|j, S) = \alpha_i u_i(\sigma|k) \leq \alpha_i x_i^{iW}$. Thus, $u_i(\sigma|j, S) \geq u_i(\sigma, 1|j, S)$.

Let $i \in N \setminus W$. Suppose that there exists $S \in \mathcal{Q}_i$ and $k \in S$ such that $\sigma_k(k) = S$ and $\sigma_l(k, S) = 1$ for any $l \in S \setminus \{k\}$. Because $|S| = |W| = q > \frac{n}{2}$, there exists $l \in S \cap W$. Because $i \in S$ and $i \notin W$, $S \neq W$. Thus, $\sigma_k(k) \neq W$. Hence, by the construction of σ, $k \notin W$. Therefore, $l \neq k$. Thus, $l \in S \setminus \{k\}$. Because $l \in W$ and $S \neq W$, $\sigma_l(k, S) = 0$, which is a contradiction. Thus, for any $S \in \mathcal{Q}_i$, there does not exist $k \in S$ such that $\sigma_k(k) = S$ and $\sigma_l(k, S) = 1$ for any $l \in S \setminus \{k\}$. Hence, for any $k \in N$ and any $S \in \mathcal{Q}_k$, $u_i(\sigma, S|k) = 0$. (i) For any $S \in \mathcal{Q}_i$, $u_i(\sigma, S|i) = 0 \leq u_i(\sigma|i)$. (ii) Let $j \in N \setminus \{i\}$ and $S \in \mathcal{Q}_j \cap \mathcal{Q}_i$. Suppose that $\sigma_i(j, S) = 1$. Then, $u_i(\sigma, 0|j, S) = \alpha_i u_i(\sigma|i) = 0 \leq u_i(\sigma|j, S)$. Suppose that $\sigma_i(j, S) = 0$. Then, there exists $k \in S \cap W \setminus \{j\}$ such that $k \triangleright i$. Because $i \in S$ and $i \notin W$, $S \neq W$. Thus, $\sigma_k(S) = 0$. Note that $k \triangleright i$. Then, there exists $l \in S$ such that $u_i(\sigma, 1|j, S) = \alpha_i u_i(\sigma|l) = 0 \leq u_i(\sigma|j, S)$.

Thus, by the one-shot deviation principle, σ is an SPE.

Characterization Let σ be an SSPE.

Lemma 5. *Let $i \in W$. Then, for any $j \in W \setminus \{i\}$, $\sigma_j(i, W) = 1$.*

Proof. By the mathematical induction, show that for any $j \in W \setminus \{i\}$, $\sigma_j(i,W) = 1$. Let $j \in W \setminus \{i\}$. Suppose that for any $k \in W \setminus \{i\}$ with $k \triangleright j$, $\sigma_k(i,W) = 1$ (induction hypothesis). Then, $u_j(\sigma, 1|i,W) = x_j^{iW} = \delta_j x_j^{jW}$. By Lemma 2, $u_j(\sigma, 0|i,W) = \alpha_j u_j(\sigma|j) \le \alpha_j x_j^{jW}$. Thus, $u_j(\sigma, 1|i,W) > u_j(\sigma, 0|i,W)$. Hence, $\sigma_j(i,W) = 1$. Therefore, for any $j \in W \setminus \{i\}$, $\sigma_j(i,W) = 1$. □

Lemma 6. *Let $i \in N \setminus W$ and $S \in \mathcal{Q}_i$. Then, for some $j \in S \setminus \{i\}$, $\sigma_j(i,S) = 0$.*

Proof. Suppose that $\sigma_j(i,S) = 1$ for any $j \in S \setminus \{i\}$. Because $|S| = |W| = q > \frac{n}{2}$, there exists $j \in S \cap W$. Because $i \notin W$, $j \neq i$. By the supposition, $u_j(\sigma|i,S) = x_j^{iS} = \delta_j x_j^{jS}$. $u_j(\sigma, 0|i,S) = \alpha_j u_j(\sigma|j)$; because σ is an SPE, $u_j(\sigma|j) \ge u_j(\sigma, W|j)$; by Lemma 5, $u_j(\sigma, W|j) = x_j^{jW}$; thus, $u_j(\sigma, 0|i,S) \ge \alpha_j x_j^{jW}$. Note that because σ is an SPE, $u_j(\sigma|i,S) \ge u_j(\sigma, 0|i,S)$. Then, $\delta_j x_j^{jS} \ge \alpha_j x_j^{jW}$. Because $S \neq W$, $\delta_j x_j^{jS} < \alpha_j x_j^{jW}$, which is a contradiction. □

Lemma 7. *Let $i \in W$. Then, $\sigma_i(i) = W$.*

Proof. By Lemma 5, $u_i(\sigma, W|i) = x_i^{iW}$. Let $S \in \mathcal{Q}_i \setminus \{W\}$. If $\sigma_j(i,S) = 1$ for any $j \in S \setminus \{i\}$, $u_i(\sigma, S|i) = x_i^{iS}$, and by Lemma 1, $u_i(\sigma, S|i) < x_i^{iW}$. Otherwise, by Lemma 2, for some $j \in S \setminus \{i\}$, $u_i(\sigma, S|i) = \alpha_i u_i(\sigma|j) \le \alpha_i x_i^{iW} < x_i^{iW}$. Thus, $u_i(\sigma, W|i) > u_i(\sigma, S|i)$. Hence, $\sigma_i(i) = W$. □

Suppose that $f \in W$. By Lemma 7, $\sigma_f(f) = W$. By Lemma 5, for any $i \in W \setminus \{f\}$, $\sigma_i(f,W) = 1$.

Suppose that $f \in N \setminus W$. Suppose that in σ, some proposal S is accepted in some period t. By Lemmas 6 and 7, $S = W$. By Lemma 6, $t \neq 1$. □

D Proof of Proposition 3

Let σ be the stationary strategy tuple such that

- $\sigma_i(i) = \{1,2\}$, and $\sigma_i(j, \{1,2\}) = \sigma_i(3, \{i,3\}) = 0$;
- $\sigma_j(j) = \{j,3\}$, and $\sigma_j(i, \{1,2\}) = 1$ and $\sigma_i(3, \{j,3\}) = 0$.

- $\sigma_3(3) = \{j,3\}$, and $\sigma_3(i,\{i,3\}) = \sigma_3(j,\{j,3\}) = 1$.

(i) $u_i(\sigma|i) = x_i^{i\{1,2\}}$. $u_i(\sigma,\{i,3\}|i) = x_i^{i\{i,3\}}$. Thus, by Lemma 1, $u_i(\sigma|i) \geq u_i(\sigma,\{i,3\}|i)$. (ii) Let $k \in N \setminus \{i\}$. $u_i(\sigma|k,\{i,k\}) = \alpha_i u_i(\sigma|i) = \alpha_i x_i^{i\{1,2\}}$. $u_i(\sigma,1|k,\{i,k\}) = x_i^{k\{i,k\}} = \delta_i x_i^{i\{i,k\}}$. Thus, because $\alpha_i > \delta_i$, by Lemma 1, $u_i(\sigma|k,\{i,k\}) \geq u_i(\sigma,1|k,\{i,k\})$.

(i) $u_j(\sigma|i) = x_j^{j\{j,3\}}$. $u_j(\sigma,\{1,2\}|j) = \alpha_j x_j^{i\{1,2\}} = \alpha_j \delta_j x_j^{j\{1,2\}}$. Thus, because $\alpha_j < \min\left\{\delta_j \frac{x_j^{j\{1,2\}}}{x_j^{j\{j,3\}}}, \frac{1}{\delta_j}\frac{x_j^{j\{j,3\}}}{x_j^{j\{1,2\}}}\right\}$, $u_j(\sigma|i) \geq u_j(\sigma,\{1,2\}|j)$. (ii) $u_j(\sigma|i,\{1,2\}) = x_j^{i\{1,2\}} = \delta_j x_j^{j\{1,2\}}$. $u_j(\sigma,0|i,\{1,2\}) = \alpha_j u_j(\sigma|j) = \alpha_j x_j^{j\{j,3\}}$. Thus, because $\alpha_j < \min\left\{\delta_j \frac{x_j^{j\{1,2\}}}{x_j^{j\{j,3\}}}, \frac{1}{\delta_j}\frac{x_j^{j\{j,3\}}}{x_j^{j\{1,2\}}}\right\}$, $u_j(\sigma|i,\{1,2\}) \geq u_j(\sigma,0|i,\{1,2\})$. $u_j(\sigma|3,\{j,3\}) = \alpha_j u_j(\sigma|j) = \alpha_j x_j^{j\{j,3\}}$. $u_j(\sigma,1|3,\{j,3\}) = x_j^{3\{j,3\}} = \delta_j x_j^{j\{j,3\}}$. Thus, because $\alpha_j > \delta_j$, $u_j(\sigma|3,\{j,3\}) \geq u_j(\sigma,1|3,\{j,3\})$.

(i) $u_3(\sigma|3) = \alpha_3 x_3^{j\{j,3\}} = \alpha_3 \delta_3 x_3^{3\{j,3\}}$. $u_3(\sigma,\{i,3\}|3) = 0$. Thus, $u_3(\sigma|3) \geq u_3(\sigma,\{i,3\}|3)$. (ii) Let $k \in \{1,2\}$. $u_3(\sigma|k,\{k,3\}) = x_3^{k\{k,3\}} = \delta_3 x_3^{3\{k,3\}}$. $u_3(\sigma,0|k,\{k,3\}) = \alpha_3 u_3(\sigma|3) = \alpha_3^2 \delta_3 x_3^{3\{j,3\}}$. Thus, because $\alpha_3 < \sqrt{\frac{x_3^{3\{2,3\}}}{x_3^{3\{1,3\}}}}$ and $x_3^{3\{2,3\}} < x_3^{3\{1,3\}}$, $u_3(\sigma|k,\{k,3\}) \geq u_3(\sigma,0|k,\{k,3\})$.

Thus, by the one-shot deviation principle, σ is an SPE. □

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