#0014

DISCUSSION PAPER

Extractive Contest Design

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Abstract

We consider contest success functions (CSFs) that extract contestants' values of the prize. In the case in which the values are observable to the contest designer, in the more-than-two-contestant or common-value subcase, we present a CSF extractive in any equilibrium; in the other subcase, we present a CSF extractive in some equilibrium, but there exists no CSF extractive in any equilibrium. In the case in which the values are not observable, there exists no CSF extractive in some equilibrium. In the case in which the values are observable and common, we present extractive a CSF extractive in any equilibrium; we present a class of CSFs extractive in some equilibrium, and this class can control the number of active contestants.

Keywords: contest success function; extraction of values; observability of values aggregate effort equivalence across equilibria

JEL classification codes: C72; D72

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1 Introduction

In this paper, we define extractiveness of contest success functions (CSFs). In a contest, which is formalized by Tullock (1980), contestants make an effort, and a winner of a prize is determined according to a probability distribution, which depends on efforts. A function that maps from effort tuples to winning probability distributions is called a contest success function (CSF). We consider the design of CSFs that extract values of the prize through contestants' efforts. We say that a CSF is *extractive* if under this CSF, there exists a Nash equilibrium such that the aggregate effort is equal to the maximum value (the maximum of contestants' values of the prize). We also say that a CSF is *strictly extractive* if this CSF is extractive, and under this CSF, in every Nash equilibrium, the aggregate effort is equal to the maximum value.

Focusing on observability of contestants' values of the prize, we present CSFs satisfying extractiveness. Firstly, we consider the case in which contestants' values are observable for the contest designer. In the subcase in which the number of players is greater than 2, or players have a common value, we present a strictly extractive CSF. In the subcase in which the number of players is 2, and the players have heterogeneous values, we present an extractive CSF, and show that there does not exist a strictly extractive CSF, even though the contest designer can fully use the information of the values. The aggregate effort equivalence between Nash equilibria holds in the former subcase but does not in the latter subcase. Secondly, we consider the case in which the values are unobservable for the contest designer. We show that there does not exist a CSF extractive under every value tuple. We consider the subcase in which players have a common value unobservable for the contest designer. We examine the CSF such that the winning probabilities are proportional to the $\frac{a}{a-1}$th powers of efforts, where a is an integer in $[2, n]$ (n is the number of contestants). We show that under every common value, this CSF is extractive, this CSF with $a = 2$ is strictly extractive, and this CSF with $a > 2$ is not strictly extractive. Thus, the aggregate effort equivalence across Nash equilibria holds under $a = 2$ but does not under $a > 2$.

Several papers have presented CSFs that are extractive but not strictly extractive in the observable-value case. In the 2-contestant common-value case, Glazer (1993) presented a CSF such that a certain contestant wins if their effort is their value, and the other contestant wins otherwise (Subsection 3.1). In the 2-contestant case (the n-contestant case, resp.), Nti (2004) (Franke et al. (2018), resp.) presented a CSF such that a contestant with the maximum value wins if their effort is greater than or equal to their value, and a contestant with the second highest value, which may be equal to the maximum value, wins otherwise (Proposition 2 (Proposition 4.7, resp.)). These CSFs are extractive. However, they are not strictly extractive, because there exists a Nash equilibrium such that every contestant's effort is zero.[1] Meanwhile, we present strictly extractive CSFs in the 3-or-more-contestant or common-value case.

Several papers have presented CSFs that are reduced to CSFs extractive in the unobservable-common-value case. In the 2-contestant case, Nti (2004) (Epstein et al. (2013); Ewerhart (2017), resp.) presented a CSF that maximizes the aggregate effort in a class of CSFs (Section 4 (Subsection 4.2; Proposition 6, resp.)). In the n-contestant common-value case, Michaels (1988) did so (Subsection 2.1). Each CSF in Nti (2004), Epstein et al. (2013) and Ewerhart (2017) in the common-value case (the CSF in Michaels (1988), resp.) is the CSF such that the winning probabilities are proportional to the 2nd ($\frac{n}{n-1}$th, resp.) powers of efforts, and it is in the unobservable-common-value case because it does not depend on the common value. The CSF using the 2nd power is strictly extractive in the 2-contestant case. We show that this CSF is also strictly extractive in the n-contestant case. The CSF using the $\frac{n}{n-1}$th power is extractive. We show that this CSF is not strictly extractive. We also show that in the n-contestant case, CSFs such that the winning probabilities are proportional to the $\frac{a}{a-1}$th powers of efforts ($2 \leq a \leq n$) is extractive. Under this CSF, any set of a contestants is the set of active contestants (i.e., contestants whose efforts are positive) in some Nash equilibrium. Thus, by selecting a and suggesting a Nash equilibrium, the contest designer can choose an arbitrary set of two-or-more active contestants. If the contest designer has a preference such that contestants' efforts are complementary with each other, they may most prefer the CSF with $a = n$.

Several papers have shown extraction of values in mixed-strategy Nash equilibria. Hillman and Riley (1989) (Baye et al. (1996), resp.) showed that in the all-pay auc-

[1] Nti (2004) suggested that if the CSF is modified as the threshold of effort is slightly lowered, the Nash equilibrium such that every contestant's effort is zero is removed. However, under this modified CSF, in a unique Nash equilibrium, the aggregate effort is slightly smaller than the maximum value.

tion, if the highest two values are equal, the expected aggregate effort is equal to the maximum value in any mixed-strategy Nash equilibrium (the second last paragraph in Section 3 (Theorem 1, resp.)). Alcalde and Dahm (2010) showed that under CSFs satisfying some conditions, there exists a mixed-strategy Nash equilibrium such that if the highest two values are equal, the expected aggregate effort is equal to the maximum value (Theorem 3.2). We show extraction of values in pure-strategy Nash equilibria.

Several papers have considered maximization of the aggregate effort in a class of CSFs. CSFs using the following devices have been examined: concave technologies and power technologies[2] in the lottery contest (Nti (2004)); power technologies in the lottery contest (Michaels (1988)); biases multiplying efforts in the lottery contest, (Franke et al. (2013)); biases multiplying efforts with power technologies in the lottery contest and biases multiplying efforts the all-pay auction auction (Epstein et al. (2013));[3] biases multiplying efforts in the lottery contest and the all-pay auction (Franke et al. (2014a)); head starts added to efforts in the lottery contest and all-pay auction (Franke et al. (2014b)); biases multiplying efforts given a power technology in the lottery contest (Ewerhart (2017)); biases multiplying efforts and head starts added to efforts in the lottery contest and the all-pay auction (Franke et al. (2018)). Fang (2002) compared the simple lottery contest and the simple all-pay auction. Owing to restriction on forms of CSFs, the maximized aggregate effort is not equal to the maximum value except for the above-mentioned results. In our paper, because no restriction is imposed on forms of CSFs, the extraction of values is achieved.

Several papers have considered the aggregate effort under asymmetric information. In Kirkegaard (2012), Pérez-Castrillo and Wettstein (2016), Matros and Possajennikov (2016), Drugov and Ryvkin (2017) and Olszewski and Siegel (2020), the values or the productivities of efforts are private information. In our paper, in the observable-value case, the contest designer and contestants know all contestants' values; in the unobservable case, the contest designer knows none of the contestants' values, but contestants know all contestants' values.

[2]A concave (power, resp.) technology is a concave (power, resp.) function that transforms efforts. The winning probabilities are determined according to the transformed efforts.

[3]Epstein et al. (2011) considered the same class of CSF but a different objective of the contest designer, which is the weighted sum of the aggregate effort and the welfare.

As well as the existing papers on maximization of the aggregate effort, our paper provides CSFs a positive foundation.[4,5] In rent-seeking interpretation, the contest designer (politician) wants to maximize contestants' efforts, because if the efforts are political contributions, they obtain monetary benefit from the efforts, and if the efforts are political lobbying, they flaunt their power by the efforts. Thus, they should determine the CSF as it maximizes the efforts. Hence, they should choose an extractive CSF if it exists.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 presents our results. Section 4 discusses questions for future research. The proofs of all propositions are provided in Appendix.

2 Model

For any sets X, Y and I, any $f : X \to Y^I$ and any $x \in X$ and $i \in I$, let $f_i(x)$ be the value $f(x)$ for i.

Let N be a finite set such that $|N| \geq 2$: N is the set of contestants. Let $n := |N|$. Let $X := \mathbb{R}^N_{\geq 0}$: X is the set of tuples of contestants' efforts. Let $\Delta := \left\{p \in \mathbb{R}^N_{\geq 0} \mid \sum_{i \in N} p_i = 1\right\}$: the set of tuples of contestants' success probabilities (for any $p \in \Delta$ and any $i \in N$, p_i is the probability of contestant i's winning). Let $F := \{f \mid f : X \to \Delta\}$: F is the set of contest success functions (CSFs). Let $V := \mathbb{R}^N_{>0}$: the set of tuples of contestants' values of the prize. For any $f \in F$ and any $v \in V$, let $u^{fv} : X \to \mathbb{R}^N$ such that for any $x \in X$ and $i \in N$, $u^{fv}_i(x) = f_i(x)\, v_i - x_i$: $u^{fv}_i(x)$ is contestant i's utility from effort tuple x ($f_i(x)\, v_i$ is the expected value that they obtain, and x_i is the cost of their effort).

For any $f \in F$ and any $v \in V$, $\left(N, X, u^{fv}\right)$ is a strategic form game: N is the set of players, X is the set of strategy tuples, and u^{fv} is the function that maps each strategy tuple to the payoff tuple by it. For any $f \in F$ and any $v \in V$, let E^{fv} be the set of Nash equilibria in $\left(N, X, u^{fv}\right)$.

Let $\hat{V} := \{v \in V \mid \forall i, j \in N\, (v_i = v_j)\}$: the set of value tuples such that all

[4] Jia et al. (2013) referred to this as the *optimally-derived foundation*, which is one of four types of foundations.

[5] Some papers have provided CSFs axiomatic foundations (e.g., Skaperdas (1996) and Clark and Riis (1998)).

contestants have a common value. For any $v \in V$, let $m^v := \max_{i \in N} v_i$ and $M^v := \arg\max_{i \in N} v_i$: m^v is the maximum of contestants' values of the prize, and M^v is the set of contestants who have the maximum value.

3 Results

First, we show that the maximum value of the prize is an upper bound of the equilibrium aggregate effort. Subsequently, we seek CSFs under which the equilibrium aggregate effort is equal to the maximum value in the case in which the values are observable to the contest designer and the case in which they are unobservable, respectively. Note that the values are observable to contestants.

3.1 Bound of aggregate effort

For any CSF and any value tuple, in any Nash equilibrium, the aggregate effort is less than or equal to the maximum value.

Proposition 1. *Let $f \in F$ and $v \in V$. Let $x^* \in E^{fv}$. Then, $\sum_{i \in N} x_i^* \leq m^v$.*

We say that a CSF is extractive if in some Nash equilibrium, the aggregate effort is equal to the maximum value. We say that a CSF is strictly extractive if it is extractive, and in any Nash equilibrium, the aggregate effort is equal to the maximum value.

Definition 1. Let $f \in F$ and $v \in V$. f is *extractive* under v if there exists $x^* \in E^{fv}$ such that $\sum_{i \in N} x_i^* = m^v$. f is *strictly extractive* under v if f is extractive under v and for all $x^* \in E^{fv}$, $\sum_{i \in N} x_i^* = m^v$.

3.2 Observable-value case

We consider the case in which the contest designer can observe contestants' values and thus, can design CSFs dependent on the values.

Under any value tuple, if the number of contestants is greater than or equal to 3, or all contestants have a common value, some CSF is strictly extractive; otherwise, some CSF is extractive, but any CSF is not strictly extractive. Thus, the aggregate effort equivalence across Nash equilibria holds in the former case but does not in the latter case.

Proposition 2. *Let $v \in V$. Suppose that $n \geq 3$. Let $f \in F$ such that for some distinct $i, j, k \in N$ such that $i \in M^v$, for any $x \in X$, (i) if $x_i = m^v$, then $f_i(x) = 1$, (ii) if $x_i \neq m^v$ and $x_j > 0$, then $f_j(x) = 1$, and (iii) if $x_i \neq m^v$ and $x_j = 0$, then $f_k(x) = 1$. Then, f is strictly extractive under v.*

Proposition 3. *Let $v \in V$. Suppose that $n = 2$ and $v \in \hat{V}$. Let $f \in F$ such that for any $i \in N$, any $j \in N \setminus \{i\}$ and any $x \in X$, if $x_k > 0$ for any $k \in N$, $f_i(x) = \frac{\mathbf{1}_{x_i = m^v} - \mathbf{1}_{x_j = m^v} + 1}{2}$; otherwise, $f_i(x) = \frac{\mathbf{1}_{x_i > 0} - \mathbf{1}_{x_j > 0} + 1}{2}$. Then, f is strictly extractive under v.*

Proposition 4. *Let $v \in V$. Suppose that $n = 2$ and $v \notin \hat{V}$. (i) Let $f \in F$ such that for some $i \in M^v$, for any $x \in X$, $f_i(x) = \mathbf{1}_{x_i = m^v}$. Then, f is extractive under v. (ii) Let $f \in F$. f is not strictly extractive under v.*

The above CSFs make the contestant with the maximum value win with certainty (or with probability $\frac{1}{2}$) if their effort is equal to the maximum value (or $\frac{1}{2}$ of the maximum value), in order that the aggregate effort is equal to the maximum value. In the 3-or-more contestant or common-value case, the above CSFs give contestants an incentive to make a positive effort, to exclude the Nash equilibrium such that every contestant's effort is zero. In the other case, under any extractive CSF, there exists a Nash equilibrium such that every contestant's effort is zero.

3.3 Unobservable-value case

We consider the case in which the contest designer cannot observe contestants' values, and design CSFs independent of the values.

For any CSF f, for some value tuple v, f is not extractive under v.

Proposition 5. *Let $f \in F$. Then, for some $v \in V$, f is not extractive under v.*

We focus on the common-value case and define candidates of CSFs (strictly) extractive under any common value. Let $f \in F$ such that for some $a \in \mathbb{N}$ such that

$2 \leq a \leq n$, for any $i \in N$ and any $x \in X$,

$$f_i(x) = \begin{cases} \frac{x_i^{\frac{a}{a-1}}}{\sum_{j \in N} x_j^{\frac{a}{a-1}}} & \text{if } \exists j \in N \, (x_j > 0) \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

f is extractive under any common value; f with $a = 2$ is strictly extractive under any common value; under any common value, f with $a \geq 3$ is not strictly extractive. Thus, the aggregate effort equivalence across Nash equilibria holds under $a = 2$ but does not under $a \geq 3$.

Proposition 6. *Let $v \in \hat{V}$. Then, f is extractive under v.*

Remark 1. As seen in the proof, for any $x \in X$, $x \in E^{fv}$ if for some $A \in 2^N$ such that $|A| = a$, for any $i \in A$, $x_i = \frac{m^v}{a}$ and for any $i \in N \setminus A$, $x_i = 0$.

Proposition 7. *Suppose that $a = 2$. Let $v \in \hat{V}$. Then, f is strictly extractive under v.*

Remark 2. As seen in the proof, for any $x \in X$, $x \in E^{fv}$ if and only if for some $A \in 2^N$ such that $|A| = 2$, for any $i \in A$, $x_i = \frac{m^v}{2}$ and for any $i \in N \setminus A$, $x_i = 0$.

Proposition 8. *Suppose that $3 \leq a \leq n$. Let $v \in \hat{V}$. Then, f is not strictly extractive under v.*

Remark 3. As seen in the proof, for any $x \in X$, $x \in E^{fv}$ if for some $A \in 2^N$ such that $|A| = a - 1$, for any $i \in A$, $x_i = \frac{m^v a(a-2)}{(a-1)^3}$ and for any $i \in N \setminus A$, $x_i = 0$. The aggregate effort in the strategy tuples is $(a-1)\frac{va(a-2)}{(a-1)^3} = \frac{va(a-2)}{(a-1)^2} < v$. Let $a \in \mathbb{N}^{\mathbb{N}}$ such that $3 \leq a_n \leq n$ and $\lim_{n \to \infty} a_n = \infty$. Then, $\lim_{n \to \infty} \frac{va_n(a_n-2)}{(a_n-1)^2} = v$.

Under f, each contestant's effort x is transformed to $x^{\frac{a}{a-1}}$, and the winning probabilities are determined proportionally to the transformed efforts. As a is larger, the elasticity of transformed effort $x^{\frac{a}{a-1}}$ to effort x, $\frac{dx^{\frac{a}{a-1}}/dx}{x^{\frac{a}{a-1}}/x} = \frac{a}{a-1}$, is smaller; thus, in the Nash equilibrium such that the aggregate effort is equal to the maximum value, each active contestant's effort is smaller, but the number of active contestants is larger.

4 Discussion

In the unobservable-value case, unless a common value is assumed, there does not exist extractive CSF. In such case, it is necessary to derive CSFs that maximize the expectation of the aggregate effort under some belief on value tuples. For example, this problem is formalized as

$$\max_{(f,x)\in F\times X^V} \int_{v\in V} \sum_{i\in N} x_i(v)\, dP(v)$$
$$\text{s.t. } \forall v \in V \left(x(v) \in E^{fv} \right) \wedge x \text{ is measurable,}$$

where P is a cumulative distribution function on V (the designer's belief on value tuples). This problem remains for future research.

Appendix

Lemma 1. *Let $f \in F$, $v \in V$, $x^* \in E^{fv}$ and $i \in N$. Then, $u_i^{fv}(x^*) \geq 0$, and $f_i(x^*) v_i \geq x_i^*$.*

Proof. Because $x^* \in E^{fv}$, $u_i^{fv}(x^*) \geq u_i^{fv}\left(0, x_{-i}^*\right) = f_i\left(0, x_{-i}^*\right) v_i \geq 0$. Thus, $f_i(x^*) v_i \geq x_i^*$. □

Lemma 2. *Let $v \in \mathbb{R}_{>0}$ and $a, b \in \mathbb{N}$ such that $2 \leq b \leq a$. Let $x^* := \frac{va(b-1)}{b^2(a-1)}$. Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for any $x \in \mathbb{R}_{\geq 0}$, $u(x) = \frac{x^{\frac{a}{a-1}}}{x^{\frac{a}{a-1}} + (b-1)(x^*)^{\frac{a}{a-1}}} v - x$. Then, $x^* \in \arg\max_{x \in \mathbb{R}_{\geq 0}} u(x)$.*

Proof. Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for any $x \in \mathbb{R}_{\geq 0}$,

$$\phi(x) = -\left((2b-1)(x^*)^{\frac{a}{a-1}} + x^{\frac{a}{a-1}}\right) \sum_{i=0}^{a-1} (x^*)^{\frac{i}{a-1}} x^{\frac{a-1-i}{a-1}} + b^2 (x^*)^{\frac{2a-1}{a-1}}.$$

For any $x \in \mathbb{R}_{\geq 0}$,

$$\frac{du(x)}{dx} = \frac{\phi(x)\left(x^{\frac{1}{a-1}} - (x^*)^{\frac{1}{a-1}}\right)}{\left(x^{\frac{a}{a-1}} + (b-1)(x^*)^{\frac{a}{a-1}}\right)^2}.$$

Note that $\phi(0) = (b-1)^2 (x^*)^{\frac{2a-1}{a-1}} > 0$, and $\phi(x^*) = -b(2a-b)(x^*)^{\frac{2a-1}{a-1}} < 0$. Then, by the intermediate value theorem, there exists $\bar{x} \in (0, x^*)$ such that $\phi(\bar{x}) = 0$. Let $x \in \mathbb{R}_{\geq 0}$. Because ϕ is strictly decreasing, $\phi(x) \gtreqless 0$ if and only if $x \lesseqgtr \bar{x}$. Thus,

$$\frac{du(x)}{dx} \begin{cases} \leq 0 & \text{if } x \leq \bar{x} \\ \geq 0 & \text{if } \bar{x} < x \leq x^* \\ < 0 & \text{if } x > x^*. \end{cases}$$

Note that $u(0) = 0 \leq u(x^*)$. Then, for any $x \in \mathbb{R}_{\geq 0}$, $u(x^*) \geq u(x)$. □

Proof of Proposition 1 By Lemma 1, for any $i \in N$, $x_i^* \leq v_i f_i(x^*) \leq m^v f_i(x^*)$. Hence, $\sum_{i \in N} x_i^* \leq m^v \sum_{i \in N} f_i(x^*) = m^v$. □

Proof of Proposition 2 Let $x^* \in X$ such that $x_i^* = m^v$ and for any $l \in N \setminus \{i\}$, $x_l^* = 0$. Then, $\sum_{l \in N} x_l^* = m^v$. It suffices to show that $E^{fv} = \{x_i^*\}$.

For any $x_i \in \mathbb{R}_{\geq 0} \setminus \{x_i^*\}$, $u_i^{fv}(x^*) = 0 \geq -x_i = u_i^{fv}(x_i, x_{-i}^*)$. For any $l \in N \setminus \{i\}$ and any $x_l \in \mathbb{R}_{\geq 0} \setminus \{x_l^*\}$, $u_l^{fv}(x^*) = 0 \geq -x_l = u_l^{fv}(x_l, x_{-l}^*)$. Thus, $x^* \in E^{fv}$.

Let $x \in X \setminus \{x^*\}$. If $x_i = m^v$, then for some $l \in N \setminus \{i\}$, $x_l > 0$, and $u_l^{fv}(x) = -x_l < 0 = u_l^{fv}(0, x_{-l})$. If $x_i \neq m^v$ and $x_j > 0$, then $u_j^{fv}(x) = v_j - x_j < v_j - \frac{x_j}{2} = u_j^{fv}\left(\frac{x_j}{2}, x_{-j}\right)$. If $x_i \neq m^v$ and $x_j = 0$, then $u_j^{fv}(x) = 0 < \frac{v_j}{2} = u_j^{fv}\left(\frac{v_j}{2}, x_{-j}\right)$. Thus, $x \notin E^{fv}$. □

Proof of Proposition 3 Let $x^* \in X$ such that for any $i \in N$, $x_i^* = \frac{m^v}{2}$. Then, $\sum_{i \in N} x_i^* = m^v$. It suffices to show that $E^{fv} = \{x^*\}$.

For any $i \in N$ and any $x_i \in \mathbb{R}_{\geq 0} \setminus \{x_i^*\}$, $u_i^{fv}(x^*) = 0 \geq -x_i = u_i^{fv}(x_i, x_{-i}^*)$. Thus, $x^* \in E^{fv}$.

Let $x \in X \setminus \{x^*\}$. If for any $i \in N$, $x_i > 0$, for some $i, j \in N$, $x_i = \frac{m^v}{2}$ and $x_j \neq \frac{m^v}{2}$, then $u_j^{fv}(x) = -x_j < 0 = u_j^{f}(0, x_{-j})$. If for any $i \in N$, $x_i > 0$ and $x_i \neq \frac{m^v}{2}$, then for some $i \in N$, $u_i^{fv}(x) = \frac{v_i}{2} - x_i < \frac{v_i}{2} = u_i^{fv}\left(\frac{m^v}{2}, x_{-i}\right)$. If for some $i, j \in N$, $x_i > 0$ and $x_j = 0$, then $u_i^{fv}(x) = v_i - x_i < v_i - \frac{x_i}{2} = u_i^{fv}\left(\frac{x_i}{2}, x_{-i}\right)$. If for any $i \in N$, $x_i = 0$, then for some $i \in N$, $u_i^{fv}(x) = \frac{v_i}{2} < \frac{3v_i}{4} = u_i^{fv}\left(\frac{v_i}{4}, x_{-i}\right)$. Thus, $x \notin E^{fv}$. □

Proof of Proposition 4 (i) Let $j \in N \setminus \{i\}$. Let $x^* \in X$ such that $x_i^* = m^v$, and $x_j^* = 0$. For any $x_i \in \mathbb{R}_{\geq 0} \setminus \{x_i^*\}$, $u_i^{fv}(x^*) = 0 \geq -x_i = u_i^{fv}(x_i, x_{-i}^*)$. For any $x_j \in \mathbb{R}_{\geq 0} \setminus \left\{x_j^*\right\}$, $u_j^{fv}(x^*) = 0 \geq -x_j = u_i^{fv}\left(x_j, x_{-j}^*\right)$. Thus, $x^* \in E^{fv}$. $\sum_{k \in N} x_k^* = m^v$.

(ii) Let $i, j \in N$ such that $v_i > v_j$. Suppose that f is extractive. Then, there exists $x^* \in E^{fv}$ such that $x_i^* + x_j^* = m^v$. By Lemma 1, $m^v = x_i^* + x_j^* \leq f_i(x^*) m^v + f_j(x^*) v_j$. Thus, $f_j(x^*)(m^v - v_j) \leq 0$. Hence, $f_j(x^*) = 0$. Thus, by Lemma 1, $x_j^* = 0$. Thus, $x_i^* = m^v$. Hence, $0 = u_i^{fv}(x^*) \geq u_i^{fv}\left(0, x_{-i}^*\right) = f_i\left(0, x_{-i}^*\right) v_i$. Thus, $f_i\left(0, x_{-i}^*\right) = 0$. Hence, $f_i(0,0) = 0$. Let $y^* \in X$ such that $y_i^* = y_j^* = 0$. Because $x^* \in E^{fv}$ and $x_j^* = y_j^*$, for any $y_i \in \mathbb{R}_{\geq 0}$, $u_i^{fv}(y^*) = f_i(0,0) v_i = 0 = u_i^{fv}(x^*) \geq u_i^{fv}\left(y_i, x_{-i}^*\right) = u_i^{fv}\left(y_i, y_{-i}^*\right)$; for any $y_j \in \mathbb{R}_{\geq 0}$, $u_j^{fv}(y^*) = f_j(0,0) v_j = v_j \geq f_j\left(y_j, y_{-j}^*\right) v_j - y_j = u_j^{fv}\left(y_j, y_{-j}^*\right)$. Thus, $y^* \in E^{fv}$. $y_i^* + y_j^* = 0 \neq m^v$. Thus, f is not strictly extractive under v. □

Proof of Proposition 5 Suppose that for any $v \in V$, there exists $x^* \in E^{fv}$ such that $\sum_{i\in N} x_i^* = m^v$ (assumption to a contradiction).

Let $v \in V$ such that for some $i \in N$, for any $j \in N \setminus \{i\}$, $v_i > v_j$. Let $x^* \in E^{fv}$ such that $\sum_{j\in N} x_j^* = v_i$. By Lemma 1, $v_i = \sum_{j\in N} x_j^* \leq \sum_{j\in N} v_j f_j(x^*)$. Thus, $f_i(x^*) = 1$, and for any $j \in N \setminus \{i\}$, $f_j(x^*) = 0$. Hence, by Lemma 1, for any $j \in N \setminus \{i\}$, $x_j^* = 0$, and $x_i^* = v_i$.

Let $v, w \in \mathbb{R}^N_{>0}$ such that for some $i \in N$, $v_i = 1$ and $w_i = 2$, and $v_j < v_i$ and $w_j < w_i$ for any $j \in N \setminus \{i\}$. Then, by the assumption to a contradiction, there exist $x^* \in E^{fv}$ and $y^* \in E^{fw}$ such that $\sum_{j\in N} x_j^* = v_i$ and $\sum_{j\in N} y_j^* = w_i$. Thus, $x_i^* = 1$ and $y_i^* = 2$, and for any $j \in N \setminus \{i\}$. $x_j^* = y_j^* = 0$. Moreover, $f_i(x^*) = f_i(y^*) = 1$. Thus, $u_i^{fw}\left(1, y_{-i}^*\right) = 2f_i\left(1, y_{-i}^*\right) - 1 = 2f_i(x^*) - 1 = 1 > 0 = u_i^{fw}(y^*)$, which contradicts that $y^* \in E^{fw}$. □

Proof of Proposition 6 Abuse v as the common value ($v = m^v = v_i$ for any $i \in N$). Let $x^* \in X$ such that for some $A \in 2^N$ such that $|A| = a$, for any $i \in A$, $x_i^* = \frac{v}{a}$ and for any $j \in N \setminus A$, $x_j^* = 0$. Let $i \in A$. By Lemma 2 with $b = a$, for any $x_i \in \mathbb{R}_{\geq 0}$, $u_i^{fv}(x^*) \geq u_i^{fv}\left(x_i, x_{-i}^*\right)$. Let $j \in N \setminus A$. For any $x_j \in \mathbb{R}_{>0}$,

$$
\begin{aligned}
u_j^{fv}\left(x_j, x_{-j}^*\right) &= \frac{x_j^{\frac{a}{a-1}}}{x_j^{\frac{a}{a-1}} + a\left(\frac{v}{a}\right)^{\frac{a}{a-1}}} v - x_j < \frac{x_j^{\frac{a}{a-1}}}{x_j^{\frac{a}{a-1}} + (a-1)\left(\frac{v}{a}\right)^{\frac{a}{a-1}}} v - x_j \\
&= u_i^{fv}\left(x_j, x_{-i}^*\right) \leq u_i^{fv}(x^*) = 0 = u_j^{fv}(x^*).
\end{aligned}
$$

Thus, $x^* \in E^{fv}$. $\sum_{i\in N} x_i^* = a \cdot \frac{v}{a} = v$. □

Proof of Proposition 7 By Proposition 6, f is extractive.

Abuse v as the common value ($v = m^v = v_i$ for any $i \in N$). Let $x^* \in E^{fv}$. Let $A := \{i \in N \mid x_i^* > 0\}$ and $\alpha := |A|$. If $\alpha = 0$, for some $i \in N$, $u_i^{fv}(x^*) = \frac{v}{n} < \frac{(2n-1)v}{2n} = u_i^{fv}\left(\frac{v}{2n}, x_{-i}^*\right)$, which contradicts that $x^* \in E^{fv}$. If $\alpha = 1$, for some $i \in A$, $u_i^{fv}(x^*) = v - x_i^* < v - \frac{x_i^*}{2} = u_i^{fv}\left(\frac{x_i^*}{2}, x_{-i}^*\right)$, which contradicts that $x^* \in E^{fv}$. Thus, $\alpha \geq 2$. Let $i \in \arg\max_{j\in A} x_j^*$. For any $k \in A$,

$$
0 = \frac{\partial u_k^{fv}}{\partial x_k}(x^*) = \frac{2v x_k^* \left(\sum_{l\in A} (x_l^*)^2 - (x_k^*)^2\right)}{\left(\sum_{l\in A} (x_l^*)^2\right)^2} - 1.
$$

Thus, for any $k \in A\backslash\{i\}$ such that $x_k^* < x_i^*$, $x_i^* \left(\sum_{l\in A} (x_l^*)^2 - (x_i^*)^2\right) = x_k^* \left(\sum_{l\in A} (x_l^*)^2 - (x_k^*)^2\right)$, $(x_i^* - x_k^*) \left(\sum_{l\in A\backslash\{i,k\}} (x_l^*)^2 - x_i^* x_k^*\right) = 0$, $\sum_{l\in A\backslash\{i,k\}} (x_l^*)^2 = x_i^* x_k^*$. Suppose that for some $j \in A \setminus \{i\}$, $x_j^* < x_i^*$ (assumption to a contradiction). Then, $\sum_{l\in A\backslash\{i,j\}} (x_l^*)^2 = x_i^* x_j^*$. For any $k \in A \setminus \{i,j\}$, $(x_k^*)^2 \leq \sum_{l\in A\backslash\{i,j\}} (x_l^*)^2 = x_i^* x_j^* < (x_i^*)^2$, and thus, $x_k^* < x_i^*$. Hence, for any $k \in A \setminus \{i\}$, $x_k^* < x_i^*$. Thus, for any $k \in A \setminus \{i\}$, $x_i^* x_j^* + \left(x_j^*\right)^2 = \sum_{l\in A\backslash\{i\}} (x_l^*)^2 = x_i^* x_k^* + (x_k^*)^2$, $\left(x_j^* - x_k^*\right)\left(x_j^* + x_k^* + x_i^*\right) = 0$, and $x_j^* = x_k^*$. Thus, $(\alpha - 2)\left(x_j^*\right)^2 = x_i^* x_j^*$, and $x_i^* = (\alpha - 2)\, x_j^*$. Hence, $0 = \frac{\partial u_j^{fv}}{\partial x_j}(x^*) = \frac{2v(\alpha-1)(\alpha-2)\left(x_j^*\right)^3}{\left((\alpha^2-3\alpha+3)\left(x_j^*\right)^2\right)^2} - 1$, $x_j^* = \frac{2v(\alpha-1)(\alpha-2)}{(\alpha^2-3\alpha+3)^2}$, and $\alpha \geq 3$. Thus, $u_j^{fv}(x^*) = \frac{\left(x_j^*\right)^2}{\left((\alpha-2)x_j^*\right)^2 + (\alpha-1)\left(x_j^*\right)^2} v - x_j^* = -\frac{v(\alpha(\alpha-3)+1)}{(\alpha^2-3\alpha+3)^2} < 0 = u_j^{fv}\left(0, x_{-j}^*\right)$, which contradicts that $x^* \in E^{fv}$. Hence, for any $j \in A$, $x_j^* = x_i^*$. Thus, $0 = \frac{\partial u_i^{fv}}{\partial x_i}(x^*) = \frac{2v(\alpha-1)}{\alpha^2 x_i^*} - 1$, and $x_i^* = \frac{2v(\alpha-1)}{\alpha^2}$. Hence, $\frac{v(2-\alpha)}{\alpha^2} = u_i^{fv}(x^*) \geq u_i^{fv}\left(0, x_{-i}^*\right) = 0$. Thus, $\alpha = 2$. Hence, $x_i^* = \frac{v}{2}$. Thus, for any $j \in A$, $x_j^* = \frac{v}{2}$. Hence, $\sum_{i\in N} x_i^* = 2 \cdot \frac{v}{2} = v$. □

Proof of Proposition 8 Abuse v as the common value ($v = m^v = v_i$ for any $i \in N$). Let x^* be a strategy tuple such that for some $A \in 2^N$ such that $|A| = a - 1$, for any $i \in A$, $x_i^* = \frac{va(a-2)}{(a-1)^3}$ and for any $j \in N \setminus A$, $x_j^* = 0$. Let $i \in A$. By Lemma 2 with $b = a - 1$, for any $x_i \in \mathbb{R}_{\geq 0}$, $u_i^{fv}(x^*) \geq u_i^{fv}\left(x_i, x_{-i}^*\right)$. Let $j \in N \setminus A$. For any $x_j \in \mathbb{R}_{>0}$,

$$u_j^{fv}\left(x_j, x_{-j}^*\right) = \frac{x_j \left(x_j^{\frac{1}{a-1}} v - x_j^{\frac{a}{a-1}} - (a-1)\left(\frac{va(a-2)}{(a-1)^3}\right)^{\frac{a}{a-1}}\right)}{x_j^{\frac{a}{a-1}} + (a-1)\left(\frac{va(a-2)}{(a-1)^3}\right)^{\frac{a}{a-1}}}$$

$$\frac{d\left(v x_j^{\frac{1}{a-1}} - x_j^{\frac{a}{a-1}}\right)}{dx_j} = \frac{a}{a-1} x_j^{\frac{2-a}{a-1}} \left(\frac{v}{a} - x_j\right).$$

Thus, for any $x_j \in \mathbb{R}_{\geq 0}$,

$$u_j^{fv}\left(x_j, x_{-j}^*\right) \leq \frac{x_j \left(\left(\frac{v}{a}\right)^{\frac{1}{a-1}} v - \left(\frac{v}{a}\right)^{\frac{a}{a-1}} - (a-1)\left(\frac{va(a-2)}{(a-1)^3}\right)^{\frac{a}{a-1}}\right)}{x_j^{\frac{a}{a-1}} + (a-1)\left(\frac{va(a-2)}{(a-1)^3}\right)^{\frac{a}{a-1}}}$$

$$= -\frac{(a-1)\, x_j \left(\frac{v}{a}\right)^{\frac{a}{a-1}} \left(\left(1 + \frac{a(a-3)+1}{(a-1)^3}\right)^{\frac{a}{a-1}} - 1\right)}{x_j^{\frac{a}{a-1}} + (a-1)\left(\frac{va(a-2)}{(a-1)^3}\right)^{\frac{a}{a-1}}} \leq 0 = u_j^{fv}(x^*).$$

Thus, $x^* \in E^{fv}$. $\sum_{i \in N} x_i^* = (a-1) \cdot \frac{va(a-2)}{(a-1)^3} = \frac{(a-1)^2-1}{(a-1)^2} v \neq v$. Thus, f is not strictly extractive under v. $\square$

References

J. Alcalde and M. Dahm. Rent seeking and rent dissipation: A neutrality result. *Journal of Public Economics*, 94:1–7, 2010.

M. R. Baye, D. Kovenock, and C. G. de Vries. The all-pay auction with complete information. *Economic Theory*, 8:291–305, 1996.

D. J. Clark and C. Riis. Contest success functions: an extension. *Economic Theory*, 11:201–204, 1998.

M. Drugov and D. Ryvkin. Biased contests for symmetric players. *Games and Economic Behavior*, 103:116–144, 2017.

G. S. Epstein, Y. Mealem, and S. Nitzan. Political culture and discrimination in contests. *Journal of Public Economics*, 95:88–93, 2011.

G. S. Epstein, Y. Mealem, and S. Nitzan. Lotteries vs. all-pay auctions in fair and biased contests. *Economics and Politics*, 25:48–60, 2013.

C. Ewerhart. Revenue ranking of optimally biased contests: The case of two players. *Economics Letters*, 157:167–170, 2017.

H. Fang. Lottery versus all-pay auction models of lobbying. *Public Choice*, 112: 351–371, 2002.

J. Franke, C. Kanzow, W. Leininger, and A. Schwartz. Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory*, 52:589–630, 2013.

J. Franke, C. Kanzow, W. Leininger, and A. Schwartz. Lottery versus all-pay auction contests: A revenue dominance theorem. *Games and Economic Behavior*, 83:116–126, 2014a.

J. Franke, W. Leininger, and C. Wasser. Revenue maximizing head starts in contests. Technical Report 524, Ruhr Economic Paper, 2014b.

J. Franke, W. Leininger, and C. Wasser. Optimal favoritism in all-pay auctions and lottery contests. *European Economic Review*, 104:22–37, 2018.

A. Glazer. On the incentives to establish and play political rent-seeking games. *Public Choice*, 75:139–148, 1993.

A. L. Hillman and J. G. Riley. Politically contestable rents and transfers. *Economics and Politics*, 1:17–39, 1989.

H. Jia, S. Skaperdas, and S. Vaidya. Contest functions: Theoretical foundations and issues in estimation. *International Journal of Industrial Organization*, 31:211–222, 2013.

R. Kirkegaard. Favoritism in asymmetric contests: Head starts and handicaps. *Games and Economic Behavior*, 76:226–248, 2012.

A. Matros and A. Possajennikov. Tullock contests may be revenue superior to auctions in a symmetric setting. *Economics Letters*, 142:74–77, 2016.

R. Michaels. The design of rent-seeking competitions. *Public Choice*, 56:17–29, 1988.

K. O. Nti. Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, 20:1059–1066, 2004.

W. Olszewski and R. Siegel. Performance-maximizing large contests. *Theoretical Economics*, 15:57–88, 2020.

D. Pérez-Castrillo and D. Wettstein. Discrimination in a model of contests with incomplete information about ability. *International Economic Review*, 57:881–914, 2016.

S. Skaperdas. Contest success functions. *Economic Theory*, 7:283–290, 1996.

G. Tullock. Efficient rent seeking. In J. M. Buchanan, R. D. Tollison, and G. Tullock, editors, *Toward a Theory of the Rent-Seeking Society*, pages 97–112. Texas A&M University Press, 1980.